Weighted projective spaces.

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Weighted projective spaces I

Let k be a field and $(q_0, \ldots, q_n) \in \mathbb{Z}^{n+1}$ a fixed tuple of positive integers called **weights**. Consider the action of $k^\star = k \setminus \{0\}$ on $\mathbb{A}^{n+1}(k)$ as follows

$$\lambda \star (x_0, \ldots, x_n) = (\lambda^{q_0} x_0, \ldots, \lambda^{q_n} x_n) \qquad \text{for } \lambda \in k^* \tag{1}$$

The quotient of this action is called a **weighted projective space** and denoted by $\mathbb{WP}^n_{(q_0,\ldots,q_n)}(k)$.
The space $\mathbb{WP}^n_w(k)$ is called **well-formed weighted projective space** if

$$\gcd(q_0, \ldots, \hat{q_i}, \ldots, q_n) = 1$$

An element $\mathfrak{p} \in \mathbb{WP}^n_w(k)$ is denoted by $\mathfrak{p} = [x_0 : x_1 : \cdots : x_n]$.

Example

The space $\mathbb{WP}^n_{(1,\ldots,1)}(k)$ is the projective space $\mathbb{P}^n(k)$.

$$[x_0, \ldots, x_n] \equiv [\lambda x_0, \ldots, \lambda x_n] \quad \textit{for } \lambda \in k^*$$

Weighted polynomial ring I

Consider the ring of polynomials $k[x_0, \dots, x_n]$ and assign to every variable x_i the weight $\mathrm{wt}(x_i) = q_i$, for all $i = 0, \dots, n$. Every polynomial is a sum of monomials $a \cdot x^m = a \cdot \prod x_i^{m_i}$ with weight

$$\mathrm{wt}(x^m) = \sum m_i q_i.$$

This form a subring of the ring $k[x_0, \dots, x_n]$ and is denoted by $k_{\mathfrak{w}}[x_0, \dots, x_n]$.

Definition

Let $f \in k_{\mathfrak{w}}[x_0, \dots, x_n]$ Then, f is called a **weighted homogenous polynomial of degree** d if each monomial in f is weighted of degree d, i.e.

$$f(x_0, \dots, x_n) = \sum_{i=1}^{m} a_i \prod_{j=0}^{n} x_j^{d_j}, \quad a_i \in k \text{ and } m \in \mathbb{N}$$

and for all $0 \leq i \leq n$, we have that $\sum_{i=1}^{n} q_i d_i = d$.

We will write $k_{\mathfrak{w}}[x_0, \dots, x_n]_d \subset k_w[x_0, \dots, x_n]$, where $\mathfrak{w} = (q_0, \dots, q_n)$, to mean the additive group of all weighted homogenous polynomials of degree d.

Proposition

Consider the weighted polynomial ring $R = k_{\mathfrak{w}}[x_0, \ldots, x_n]$ *, where* $q_0, \ldots, q_n$ *are positive integers such that the weight of* x_i *is* q_i *and* $d = gcd(q_0, \ldots, q_n)$*. Because* $R^{[d]} = R$ *Then,*

$$\mathbb{WP}_n(q_0, \ldots, q_n)(R) = \mathbb{WP}_n(\frac{q_0}{d}, \ldots, \frac{q_n}{d})(R^{[d]})$$

Moreover, if we suppose that $q_0, \cdots, q_n$ *have no common factor, and that* d *is a common factor of all* q_i *for* $i \neq j$ *(and therefore coprime to* q_j*). Then the* d^{th} *truncation of* R *is the polynomial ring*

$$R^{[d]} = k[x_0, \cdots, x_{j-1}, x_j^d, x_{j+1}, \cdots, x_n]$$

Thus, in this case

$$\mathbb{WP}_n(q_0, \ldots, q_n)(R) = \mathbb{WP}_n(\frac{q_0}{d}, \ldots \frac{q_{j-1}}{d}, q_j, \frac{q_{j+1}}{d}, \ldots, \frac{q_n}{d})(R^{[d]})$$

In particular by passing to a truncation $R^{[d]}$ of R, we can always write any weighted projective space as a well formed weighted projective space.

Weighted homogenous ideal I

Definition

An ideal $I \subset k_{\mathfrak{w}}[x_0, \dots, x_n]$ is called a **weighted homogenous ideal** if every element of $f \in I$ can be written as

$$f = \sum_{i=0}^{\deg f} f_i$$

where $f_i \in k_w[x_0, \dots, x_n]_i \cap I$.

Example

Let $q = (1, 3, 2, 4)$, in $k_{\mathfrak{w}}[x_0, x_1, x_2, x_3]$

$$I = (x_0^3 + x_1;\ x_0^3 \cdot x_2^3 + x_1^3;\ x_3 + x_2^2)$$

is an weighted homogenous ideal, but

$$J = (x_0^2;\ x_2^3 + x_3^2;\ x_3^3 + x_1^2) \text{ isn't, since } \deg(x_2^3) \neq \deg(x_3^2)$$

Weighted homogenous ideals I

Sum and the product of two homogenous ideals

The **sum** of two weighted homogenous ideals I and J, is denoted by $I+J$ and is defined to be

$$I + J = \{f + g | f \in I, g \in J.\}$$

The **product** of two weighted homogenous ideals I and J is denoted by IJ and is defined to be the ideal generated by

$$IJ = \langle \{ fg \,|\, f \in I, g \in J \} \rangle.$$

Proposition

If I and J are weighted homogenous ideals in $k_w[x_0, \ldots, x_n]$, then $I + J$ and IJ are weighted homogenous ideals in $k_w[x_0, \ldots, x_n]$.

Weighted projective varieties I

If $f \in k_{\mathfrak{w}}[x_0, \ldots, x_n]_d$ then for all points $P = (\lambda^{q_0} a_0, \ldots, \lambda^{q_n} a_n) \in \mathbb{WP}^n_w$:

$$f([\lambda^{q_0} \cdot x_0, \ldots, \lambda^{q_n} \cdot x_n]) = \lambda^d \cdot f([x_0, \ldots, x_n])$$

therefore for any $\lambda \in k^*$:

$$f(\lambda^{q_0} a_0, \ldots, \lambda^{q_n} a_n) = f(a_0, \ldots, a_n) \text{ if and only if } f(a_0, \ldots, a_n) = 0$$

Thus it is well defined to write $f(P) = 0$ for some $f(x_0, \ldots, x_n) \in k_{\mathfrak{w}}[x_0, \ldots, x_n]_d$ and $P \in \mathbb{WP}^n_w$. [1]

For any given weighted homogenous ideal I, we define the **weighted projective variety of** I by

$$V(I) = \left\{ \mathfrak{p} \in \mathbb{WP}^n_w \;\middle|\; f(\mathfrak{p}) = 0 \text{ for all } f \in I \right\}$$

Weighted projective varieties properties

Proposition

Let I and J be weighted homogenous ideals. Then the following hold:
i) $V(I) \cap V(J) = V(I + J)$
ii) $V(I) \cup V(J) = V(IJ)$
iii) $\mathbb{WP}^n_{\mathfrak{w}}(k) = V(0)$

Proposition

i) A finite union of weighted projective varieties is a weighted projective variety.
ii) An arbitrary intersection of weighted projective varieties is a weighted projective variety.

Conversely, given any $V \subset \mathbb{WP}^n_w$ the **weighted homogenous ideal associated to** V is given by

$$I(V) = \left\{ f \in k_{\mathfrak{w}}[x_0, \dots, x_n] \;\middle|\; f(\mathfrak{p}) = 0 \text{ for all } \mathfrak{p} \in V \right\}$$

Radical weighted homogenous ideal

Definition

A weighted homogenous ideal I is called a **radical weighted homogenous ideal** if for every $f \in k_{\mathfrak{w}}[x_0, \dots, x_n]$ such that $f^m \in I$ then $f \in I$.

Proposition

Let $V \subset \mathbb{WP}^n_{\mathfrak{w}}$ be a weighted projective variety. Then, the weighted homogenous ideal $I(V)$ associated to V is a radical weighted homogenous ideal.

Proof.

Let f and g be two polynomials in $I(V)$. Then, $f(P) = g(P) = 0$ for all points $P \in V$, i.e. they both vanish at all points P in the variety V then so does $f + g$ and fh where h is any polynomial in $I(V)$. Therefore, $I(V)$ is an weighted homogenous ideal.

Since, $k_{\mathfrak{w}}[x_0, \dots, x_n]$ is Noetherian then $I(V)$ is finitely generated, say $I(V) = \langle f_1, \dots, f_n \rangle$. However, $f_i \in k_{\mathfrak{w}}[x_0, \dots, x_n]$ for all i and therefore every f_i is weighted homogenous polynomial. Hence $I(V)$ is weighted homogenous ideal since it is generated by finitely many weighted homogenous polynomials.

Finally let us prove that $I(V)$ is radical. Let $f^r \in I(V)$. Then, for all points $P \in V$ we have that $f^r(P) = 0$. But since $f \in k_{\mathfrak{w}}[x_0, \dots, x_n]$, which is an integral domain, then $f^r(P) = (f(P))^r = 0$ implies that $f(P) = 0$ for all $P \in V$, therefore, $f \in I(V)$. Hence $I(V)$ is a radical. □

Zariski topology

Definition

The **Zariski closure** of a subset S of a weighted projective space $\mathbb{WP}^n_{\mathfrak{w}}$ is the smallest weighted projective variety that contains S.

Proposition

Let S be a subset in the weighted projective space $\mathbb{WP}^n_{\mathfrak{w}}$. Then, $V(I(S))$ is the Zariski closure of S.

So, we may define a topology on Weighted projective space, called the Zariski topology.

Example

Let $f = f(x, y, z)$ be a degree d weighted homogenous polynomial, where the weights of x, y, and z are respectively q_0, q_1, q_2. Then,

$$V(f) \subset \mathbb{WP}^2_{\mathfrak{w}}$$

is a degree d-plane curve in the weighted projective space $\mathbb{WP}^2_{(q_0,q_1,q_2)}$.

Veronese embedding

By means of Veronese embedding we could embed the same weighted projective variety in different projective spaces.
For some large enough N we can embed a weighted projective space $\mathbb{WP}_{\mathfrak{w}}$ into a “straight ” projective space $\mathbb{P}^m$. Let $q = q_0\, q_1 \cdots q_n$, The **Veronese map** is called the map

$$
\begin{aligned}
f : \mathbb{WP}^n(k) &\longrightarrow \mathbb{P}^m(k), \\
f(x_0, \dots, x_n) &= [x_0^{q/q_0}, x_0^{r_0} \cdot x_1^{r_1}, \dots, x_0^{r_0} \cdot x_1^{r_1}, \cdots x_n^{r_n}, x_1^{q/q_1}, x_1^{r_1} \cdot x_2^{r_2}, \dots, x_n^{q/q_n}]
\end{aligned}
\tag{2}
$$

where r_i satisfy the condition: $\sum_{k=0}^{n} q_i \cdot r_i = q$.
This map embeds $\mathbb{WP}_{\mathfrak{w}}^n(k)$ to a projective space $\mathbb{P}^m(k)$.

Lemma
The Veronese map is a well-defined morphism.

See [?reid] or [2, Prop. 3] for details.

Example
Take $q = (1, 1, 2)$ *and* $\mathbb{WP}^2(k)$ *then* $f(x_0, x_1, x_2) = [x_0^2, x_0 \cdot x_1, x_1^2, x_2]$ *which is a quadratic cone in* $\mathbb{P}^3(k)$.

Example

Consider the weighted projective moduli space of genus 2 curves, say $\mathbb{WP}^3_{\mathfrak{w}}(k)$ for $\mathfrak{w} = (2,4,6,10)$. Then we have that

$$\mathbb{WP}^3_{\mathfrak{w}}(k) \approx \mathbb{WP}^3_{(1,2,3,5)}(k).$$

Then $q = 2 \cdot 3 \cdot 5 = 30$ and the Veronese embedding is
$[J_2, J_4, J_6, J_{10}] \longrightarrow$
$[J_2^{30}, J_2^6 \cdot J_4^{12}, J_2^{12} \cdot J_6^6, J_2^{15} \cdot J_{10}^3, J_2^4 \cdot J_4^4 \cdot J_6^6, J_2^3 \cdot J_4^6 \cdot J_{10}^3, J_2^3 \cdot J_6^3 \cdot J_{10}^3, J_4^{15}, J_4^6 \cdot J_6^6, J_4^5 \cdot J_{10}^4, J_4^3 \cdot J_6^3 \cdot J_{10}^3, J_{10}^6]$
Since $J_{10} \neq 0$, because it is the discriminant, then we can "factorize" J_{10}^6 and we will get

$$P = \left[\frac{J_2^{30}}{J_{10}^6}, \frac{J_2^6 \cdot J_4^{12}}{J_{10}^6}, \frac{J_2^{12} \cdot J_6^6}{J_{10}^6}, \frac{J_2^{15}}{J_{10}^3}, \frac{J_2^3 \cdot J_4^6}{J_{10}^3}, \frac{J_2^6 \cdot J_6^3}{J_{10}^3}, \frac{J_2^4 \cdot J_4^4 \cdot J_6^6}{J_{10}^6}, \frac{J_4^{15}}{J_{10}^6}, \frac{J_4^6 \cdot J_6^6}{J_{10}^6}, \frac{J_4^5}{J_{10}^2}, \frac{J_4^3 \cdot J_6^3}{J_{10}^3}, \frac{J_6^{10}}{J_{10}^6}, \frac{J_6^5}{J_{10}^3}, 1 \right]$$

Thus two genus two curves C and C' are isomorphic if and only if

$$\left[\frac{J_2^{15}}{J_{10}^3}(C), \frac{J_2^3 \cdot J_4^6}{J_{10}^3}(C), \frac{J_2^6 \cdot J_6^3}{J_{10}^3}(C), \frac{J_2^4 \cdot J_4^4 \cdot J_6^6}{J_{10}^6}(C), \frac{J_4^5}{J_{10}^2}(C), \frac{J_4^3 \cdot J_6^3}{J_{10}^3}(C), \frac{J_6^5}{J_{10}^3}(C) \right]$$

$$= \left[\frac{J_2^{15}}{J_{10}^3}(C'), \frac{J_2^3 \cdot J_4^6}{J_{10}^3}(C'), \frac{J_2^6 \cdot J_6^3}{J_{10}^3}(C'), \frac{J_2^4 \cdot J_4^4 \cdot J_6^6}{J_{10}^6}(C'), \frac{J_4^5}{J_{10}^2}(C'), \frac{J_4^3 \cdot J_6^3}{J_{10}^3}(C'), \frac{J_6^5}{J_{10}^3}(C') \right]$$

□

Heights on weighted projective spaces

Let $\mathfrak{w} = (q_0, \dots, q_n)$ be a set of weights and $\mathbb{WP}^n_{\mathfrak{w}}(k)$ the weighted projective space over a number field k.
Let k be an algebraic number field, the local degree $n_v = [k_v : Q_v]$ and M_k be the set of all absolute values in k. [3]

Definition

The **weighted multiplicative height** of $\mathfrak{p} \in \mathbb{WP}^n(\bar{k})$ is defined as

$$\mathfrak{h}(\mathfrak{p}) := \prod_{v \in M_k} \max \left\{ |x_0|_v^{\frac{n_v}{q_0}}, \dots, |x_n|_v^{\frac{n_v}{q_n}} \right\} \tag{3}$$

and the **logarithmic weighted height** as

$$\log \mathfrak{h}(\mathfrak{p}) := \log \mathfrak{h}_k(\mathfrak{p}) = \sum_{v \in M_k} \max_{0 \le j \le n} \left\{ \frac{n_v}{q_j} \cdot \log |x_j|_v \right\}. \tag{4}$$

Proposition

The following are true:
i) $\mathfrak{h}_k(\mathfrak{p})$ *does not depend on the choice of coordinates of* $\mathfrak{p}$*.*
ii) $\mathfrak{h}_k(\mathfrak{p}) \geq 1$*.*

Proof.
Let $\mathfrak{p} = [x_0, \dots, x_n] \in \mathbb{WP}^n(k)$. Since $\mathfrak{p}$ is a point in the weighted projective space, any other choice of homogenous coordinates for p has the form $[\lambda^{q_0} \cdot x_0, \dots, \lambda^{q_n} \cdot x_n]$, where $\lambda \in l^*$. Then

$$h_k([\lambda^{q_0} \cdot x_0, \dots, \lambda^{q_n} \cdot x_n]) = \prod_{\nu \in M_K} \max_{0 \leq i \leq n} |\lambda^{q_i} \cdot x_i|^{n_\nu / q_i} = \prod_{\nu \in M_K} \lambda^{n_\nu} \cdot \max_{0 \leq i \leq n} |x_i|^{n_\nu / q_i}$$

$$= \prod_{\nu \in M_K} \lambda^{n_\nu} \cdot \prod_{\nu \in M_K} \max_{0 \leq i \leq n} |x_i|^{n_\nu / q_i} = h_k(P)$$

by the product formula.
(ii) We can choose the point P such that $x_i = 1$ then $h_k(P) \geq 1$. □

Weil heights on Weighted projective varieties I

Let $\mathfrak{w} = (q_0, \dots, q_n)$ be the set weights, and let $q = q_0 q_1 \cdots q_n$ be their product. We are also able to consider the weighted heights through the Weil height, via the map:[2, Prop. 3]

$$\begin{aligned} f : \mathbb{WP}^n(k) &\longrightarrow \mathbb{P}^m(k), \\ f(x_0, \dots, x_n) &= [x_0^{q/q_0}, x_1^{q/q_1}, \dots, x_n^{q/q_n}]. \end{aligned} \tag{5}$$

Lemma

The map f defined above is a well defined morphism which satisfies:

$$\mathfrak{h}_k(\mathfrak{p}) := H\left((f(\mathfrak{p}))^{\frac{1}{q}} \right) \tag{6}$$

Weil heights on Weighted projective varieties II

Proof.

$$\mathfrak{h}_k(\ p) := \prod (max\{|\ x_i\ |^{\frac{n_v}{q_i}}\})$$

$$(\mathfrak{h}_k(\mathfrak{p}))^q = \prod (max\{|\ x_i\ |^{\frac{n_v}{q_i}}\})^q = \prod max(\{|\ x_i\ |^{\frac{1}{q_i}}\}^q = \prod max\{|\ x_i\ |^{\frac{q}{q_i}}\} = H\left(f(\mathfrak{p})\right)$$

Therefore,

$$\mathfrak{h}_k(\mathfrak{p}) = H\left(f(\mathfrak{p})^{\frac{1}{q}}\right)$$

Moreover, the following holds for logarithmic heights:

$$\log \mathfrak{h}_k(\mathfrak{p}) := \frac{1}{q} \log H\left(f(\mathfrak{p})\right) \tag{7}$$

□

Lemma

The height is invariant under Galois conjugation. So, for every $\mathfrak{p} \in \mathbb{WP}^n_{\mathfrak{w}}(\bar{Q})$*, and every* $\sigma \in G_{\mathbb{Q}}$ *we have*

$$\mathfrak{h}(\mathfrak{p}^\sigma) = \mathfrak{h}(\mathfrak{p}).$$

Proof.
Let $p = [x_0, \dots, x_n] \in \mathbb{WP}^n_{\mathfrak{w}}(\bar{Q})$ Let K be a finite Galois extension of $\mathbb{Q}$ such that $p \in \mathbb{WP}^n_{\mathfrak{w}}(K)$. Let $\sigma \in G_{\mathbb{Q}}$. Then σ gives an isomorphism

$$\sigma : K \to K^\sigma$$

and also identifies the sets M_K, and M_{K^σ} as follows

$$\sigma : M_K \to M_{K^\sigma} :$$

$$\nu \to \nu^\sigma$$

Hence, for every$x \in K$ and $\nu \in M_K$, we have $|x|_\nu = |x^\sigma|_{\nu^sigma}$. Obviously σ gives as well an isomorphism:

$$\sigma : K_\nu \to K^\sigma_{\nu^\sigma}$$

Therefore $n\nu = n_{\nu^\sigma}$, where $n_{\nu^\sigma} = [K^\sigma_{\nu^\sigma} : \mathbb{Q}_\nu]$. Then

$$h_{K^\sigma}(P^\sigma) = \prod_{w \in M_{K^\sigma}} \max_{0 \le i \le n} \{|x_i^\sigma|^{n_w/q_i}\} = \prod_{\nu \in M_K} \max_{0 \le i \le n} \{|x_i^\sigma|^{n_{\nu^\sigma}/q_i}\} = \prod_{\nu \in M_K} \max_{0 \le i \le n} \{|x_i|^{n_\nu/q_i}\} = h_K(P)$$

□

Theorem

Let c_0 and d_0 be constants and $\mathbb{WP}^n_{\mathfrak{w}}(\bar{\mathbb{Q}})$ the weighted projective space. The set

$$\{\mathfrak{p} \in \mathbb{WP}^n_{\mathfrak{w}}(\bar{\mathbb{Q}}) \, : \, \mathfrak{h}_{\mathbb{Q}}(\mathfrak{p}) \leq c_0 \ \text{ and } \ [\mathbb{Q}(\mathfrak{p}) : \mathbb{Q}] \leq d_0\}$$

contains only finitely only many points.

Proof.

The proof is a direct consequence of Northcott's theorem for projective spaces Let$q = q_0 \cdot q_1 \cdots q_n$ and consider the map $\Phi \ : \mathbb{WP}_n(k) \to P_n(k)$. From Northcott's theorem for projective spaces we have that if $C_0 = c_0^{q_0}$ and $d_0 = \frac{d_0}{q}$ and $P_n(Q)$ a projective space, then the set

$$\{\Phi(\mathfrak{p}) \in P_n(Q) : H(\Phi(\mathfrak{p})) \leq C_0 \text{ and } [\mathbb{Q}(\Phi(\mathfrak{p})) : \mathbb{Q}] \leq D_0\}$$

contains only finitely many points $\Phi(\mathfrak{p})$. So we will have that $h_{\mathbb{Q}}(\mathfrak{p}) = H_{\mathbb{Q}}(\Phi(\mathfrak{p})) \leq C_0^{\frac{1}{q_0}} = c_0$.
Also, $[\mathbb{Q}(\mathfrak{p}) : \mathbb{Q}] \leq q \cdot [\mathbb{Q}(\Phi(\mathfrak{p})) : \mathbb{Q}] \leq q \cdot D_0 = d_0$.
Since Φ is a finite degree map, then are only finitely many points $\mathfrak{p} \in \mathbb{WP}^n_{\mathfrak{w}}(\bar{\mathbb{Q}})$ satisfying the above conditions. □

Definition

Let $\mathcal{X}$ be a weighted projective variety over $\bar{k}$ and $\varphi : \mathcal{X} \to \mathbb{WP}^n(\bar{k})$ a morphism. The **weighted Weil height** of $\mathfrak{p} \in \mathcal{X}(\bar{k})$, relative to φ, is defined as

$$wh_{\varphi}(\mathfrak{p}) := \mathfrak{h}(\varphi(\mathfrak{p}))$$

Local heights on projective varieties

Let k be a field, because we have an analytic structure in the weighted projective spaces, we can define in a similar way as local heights in projective varieties, the local heights on weighted projective varieties.
Let $\mathcal{X}$ be an weighted projective variety, which we assume to be irreducible. [4] We can define the Cartier divisor on $\mathcal{X}$.

Definition

A **Cartier divisor** on $\mathcal{X}$ is an equivalence class of collection of pairs $(U_i, f_i)_{i\in I}$ satisfying the following conditions:[?hindry-silv]

1. The U_i are open sets that cover $\mathcal{X}$.
2. The f_i are non zero rational functions, $f_i \in k(U_i)^* = k_{\mathfrak{w}}(\mathcal{X})^*$.
3. $\frac{f_i}{f_j} \in \mathcal{O}(U_i \cap U_j)^*$, so $\frac{f_i}{f_j}$ has no poles or zeros on $U_i \cap U_j$.

Let D be a Cartier divisor on $\mathcal{X}$. We can find effective Cartier divisors $E_1, E_2, \ldots, E_n$ and $F_1, F_2, \ldots, F_m$ that satisfy the following properties: [4]

1. $\cap_{i=1}^{n} \text{supp} E_i = \emptyset$
2. $\cap_{i=1}^{n} \text{supp} F_j = \emptyset$
3. $D + E_i$ equivalent to F_j for $i = 1, \ldots, n$ and $j = 1, \ldots, m$

Choose rational functions $f_i j$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ such that

$$\div(f_{ij}) = F_j - E_i - D$$

Weighted Heights on aweighted projective variety I

Definition

For $P \in \mathcal{X} \setminus \operatorname{supp}(D)$ and $\nu \in M_k$, we define **the local height** as

$$\lambda_{\mathcal{D},\nu}(P) := \max_{1 \leq j \leq m} \min_{1 \leq i \leq n} \{\log |f_{ij}(P)|_\nu\}$$

Definition

For $P \in \mathcal{X} \setminus \operatorname{supp}(D)$ and $\nu \in M_k$, we define **Global Weighted Heights** as

$$\lambda_{\mathcal{D},\nu}(P) := \sum_{\nu \in M_k} \max_{1 \leq j \leq m} \min_{1 \leq i \leq n} \{\log |f_{ij}(P)|_\nu\}$$

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